

HOMOGENEOUS BETA-TYPE FUNCTIONS

MARTIN HIMMEL AND JANUSZ MATKOWSKI

ABSTRACT. All beta-type functions, i.e. the functions $B_f : (0, \infty)^2 \rightarrow (0, \infty)$ of the form

$$B_f(x, y) = \frac{f(x)f(y)}{f(x+y)}$$

for some $f : (0, \infty) \rightarrow (0, \infty)$, which are p -homogeneous, are determined. Applying this result, we show that a beta-type function is a homogeneous mean iff it is the harmonic one. A reformulation of a result due to Heuvers in terms of a Cauchy difference and the harmonic mean is given.

1. INTRODUCTION

For a given $f : (0, \infty) \rightarrow (0, \infty)$, the function $B_f : (0, \infty)^2 \rightarrow (0, \infty)$ defined by

$$B_f(x, y) = \frac{f(x)f(y)}{f(x+y)}, \quad x, y > 0,$$

is called the *beta-type function*, and f is called its *generator* ([7]). The notion the beta-type function arises from the well-known relation between the Euler Beta function $B : (0, \infty)^2 \rightarrow (0, \infty)$ and the Euler Gamma function $\Gamma : (0, \infty) \rightarrow (0, \infty)$

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y > 0.$$

Given $p \in \mathbb{R}$, we examine when the beta-type function B_f is p -homogeneous, i.e. when

$$B_f(tx, ty) = t^p B_f(x, y), \quad x, y > 0.$$

Theorem 1, the main result, says that, under some regularity assumptions of the generator f , the beta-type function is p -homogeneous if, and only if, there exist $a, b > 0$ such that $f(x) = bxa^x$ for all $x > 0$. As a corollary we obtain that a beta-type function is a homogeneous pre-mean if, and only if, there exists $a > 0$ such that $f(x) = 2xa^x$ for all $x > 0$, or, equivalently, that B_f is the harmonic mean, that is $B_f = H$, where

$$H(x, y) = \frac{2xy}{x+y}, \quad x, y > 0.$$

A related companion of the beta-type function is the Cauchy difference $C_g : (0, \infty)^2 \rightarrow \mathbb{R}$ defined by

$$C_g(x, y) = g(x+y) - g(x) - g(y)$$

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for a function $g : (0, \infty) \rightarrow \mathbb{R}$. The relationship

$$B_f = \exp \circ (-C_{\log \circ f})$$

allows to reformulate Theorem 1 in terms of logarithmical homogeneity of the Cauchy difference (Corollary 3).

At the end we remark that Heuvers result [4] on a characterization of logarithmic functions can be reformulated in terms of the Cauchy difference and the harmonic mean.

2. MAIN RESULT

Theorem 1. *Let a function $f : (0, \infty) \rightarrow (0, \infty)$ be continuous or Lebesgue measurable. Then the following conditions are equivalent:*

(i) *the beta-type function B_f is p -homogeneous, i.e.*

$$B_f(tx, ty) = t^p B_f(x, y), \quad x, y, t > 0;$$

(ii) *there exist $a, b \in (0, \infty)$ such that*

$$f(x) = bxa^x, \quad x > 0$$

and

$$B_f(x, y) = b \left(\frac{xy}{x+y} \right)^p, \quad x, y > 0.$$

Proof. Assume (i) holds. Hence, by the definition of B_f , we have

$$(2.1) \quad \frac{f(tx)f(ty)}{f(t(x+y))} = t^p \frac{f(x)f(y)}{f(x+y)}, \quad x, y, t > 0,$$

which can be written in the form

$$(2.2) \quad \frac{f(t(x+y))}{t^p f(x+y)} = \frac{f(tx)}{t^p f(x)} \frac{f(ty)}{t^p f(y)}, \quad x, y, t > 0.$$

For every fixed $t > 0$ define $\varphi_t : (0, \infty) \rightarrow (0, \infty)$ by

$$\varphi_t(x) := \frac{f(tx)}{t^p f(x)}, \quad x > 0.$$

Thus, from (2.2), for arbitrary fixed $t > 0$, it holds

$$\varphi_t(x+y) = \varphi_t(x)\varphi_t(y), \quad x, y > 0,$$

stating that φ_t is an exponential function. Hence (see, for instance, [1] p. 39), for every $t > 0$, there exists a unique additive function $\alpha_t : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\varphi_t(x) = e^{\alpha_t(x)}, \quad x > 0.$$

From the definition of φ_t , we have

$$e^{\alpha_t(x)} t^p f(x) = f(tx), \quad x > 0.$$

Since the right hand side is symmetric in x and t , so is the left hand side; thus

$$e^{\alpha_x(t)} x^p f(t) = f(xt) = f(tx) = e^{\alpha_t(x)} t^p f(x), \quad x, t > 0.$$

Setting here $t = 1$ gives

$$e^{\alpha_1(x)} f(x) = f(x) = e^{\alpha_x(1)} x^p f(1), \quad x > 0,$$

and as, by assumption, f is positive, it follows that

$$\alpha_1(x) = 0, \quad x > 0.$$

and, consequently,

$$f(x) = f(1) x^p e^{\alpha_x(1)}, \quad x > 0.$$

Putting, for convenience, $\lambda : (0, \infty) \rightarrow \mathbb{R}$,

$$\lambda(x) := \alpha_x(1), \quad x > 0,$$

we have

$$(2.3) \quad f(x) = f(1) x^p e^{\lambda(x)}, \quad x > 0.$$

Inserting this into (2.1), we obtain,

$$\frac{f(1) (tx)^p e^{\lambda(tx)} f(1) (ty)^p e^{\lambda(ty)}}{f(1) [t(x+y)]^p e^{\lambda(t(x+y))}} = t^p \frac{f(1) x^p e^{\lambda(x)} f(1) y^p e^{\lambda(y)}}{f(1) (x+y)^p e^{\lambda(x+y)}}, \quad x, y, t > 0,$$

that reduces to

$$e^{\lambda(tx) + \lambda(ty) - \lambda(t(x+y))} = e^{\lambda(x) + \lambda(y) - \lambda(x+y)}, \quad x, y, t > 0,$$

whence

$$\lambda(tx) + \lambda(ty) - \lambda(t(x+y)) = \lambda(x) + \lambda(y) - \lambda(x+y), \quad x, y, t > 0.$$

Writing this in the form

$$\lambda(t(x+y)) - \lambda(x+y) = [\lambda(tx) - \lambda(x)] + [\lambda(ty) - \lambda(y)], \quad x, y, t > 0,$$

we conclude that, for any $t > 0$, the function $\omega = \omega_t : (0, \infty) \rightarrow \mathbb{R}$, defined by

$$(2.4) \quad \omega(x) := \lambda(tx) - \lambda(x), \quad x > 0,$$

is additive. From (2.3) and the assumed regularity of f we get that ω is continuous or Lebesgue measurable. Thus, ω , being additive and continuous or measurable, is of the form ([6], p. 129, see also [1])

$$\omega(x) = \omega(1)x, \quad x > 0,$$

and hence, by (2.4),

$$\lambda(tx) - \lambda(x) = (\lambda(t) - \lambda(1))x, \quad x, t > 0,$$

whence

$$\lambda(tx) = \lambda(x) + (\lambda(t) - \lambda(1))x, \quad x, t > 0.$$

The symmetry in t and x of the left hand side implies that

$$\lambda(x) + (\lambda(t) - \lambda(1))x = \lambda(t) + (\lambda(x) - \lambda(1))t, \quad x, t > 0,$$

whence

$$\lambda(x)(1-t) + \lambda(1)t = \lambda(t)(1-x) + \lambda(1)x, \quad x, t > 0.$$

Subtracting $\lambda(1)$ from both sides yields

$$\lambda(x)(1-t) + \lambda(1)t - \lambda(1) = \lambda(t)(1-x) + \lambda(1)x - \lambda(1), \quad x, t > 0,$$

whence

$$\lambda(x)(1-t) - \lambda(1)(1-t) = \lambda(t)(1-x) - \lambda(1)(1-x), \quad x, t > 0,$$

and, consequently,

$$\frac{\lambda(x) - \lambda(1)}{1-x} = \frac{\lambda(t) - \lambda(1)}{1-t}, \quad x, t > 0, \quad x \neq 1 \neq y.$$

It follows that there exists $c \in \mathbb{R}$ such that

$$\frac{\lambda(x) - \lambda(1)}{1 - x} = -c, \quad x > 0, x \neq 1,$$

whence,

$$\lambda(x) = c(x - 1) + \lambda(1), \quad x > 0,$$

and we obtain

$$\lambda(x) = cx + d, \quad x > 0,$$

where $d := \lambda(1) - c$. Inserting this function λ into (2.3), we obtain

$$f(x) = f(1) e^d x^p (e^c)^x, \quad x > 0,$$

whence, setting

$$a := e^c, \quad b := f(1) e^d,$$

we get

$$f(x) = bx^p a^x, \quad x > 0,$$

and

$$B_f(x, y) = b \left(\frac{xy}{x + y} \right)^p, \quad x, y > 0,$$

which proves (ii). The implication (ii) \implies (i) is obvious. \square

3. APPLICATIONS TO PRE-MEANS

Definition 1. Let $I \subseteq \mathbb{R}$ be an interval and $M : I^2 \rightarrow \mathbb{R}$. The M is reflexive, if

$$M(x, x) = x, \quad x \in I;$$

M is called a pre-mean in I ([8]), if it is reflexive and $M(I^2) \subseteq I$;

M is called a mean in I , if

$$\min(x, y) \leq M(x, y) \leq \max(x, y), \quad x, y \in I.$$

Remark 1. If $M : I^2 \rightarrow \mathbb{R}$ is reflexive, then $I \subseteq M(I^2)$; so a reflexive function is a pre-mean if, and only if, $M(I^2) = I$.

Remark 2. Obviously, every mean is a pre-mean, but, in general, not vice versa. Indeed, the function $M : (0, \infty)^2 \rightarrow (0, \infty)$ defined by

$$M(x, y) = \frac{2x^2 + y^2}{x + 2y}$$

is a pre-mean. Since $M(2, 1) = 3 \notin [2, 1]$ the function is not a mean. So M is not increasing in both variables because, otherwise, it would be a mean.

Remark 3. If $M : (0, \infty)^2 \rightarrow \mathbb{R}$ is reflexive and, for some $p \in \mathbb{R}$, p -homogenous, then $p = 1$.

Corollary 1. Let $f : (0, \infty) \rightarrow (0, \infty)$ be a continuous function. Then the following conditions are equivalent:

- (i) the beta-type function B_f is a homogeneous pre-mean;
- (ii) there exists $a \in (0, \infty)$ such that

$$(3.1) \quad f(x) = 2xa^x, \quad x > 0;$$

- (iii) the beta-type function coincides with the harmonic mean, i.e.

$$B_f(x, y) = \frac{2xy}{x + y}, \quad x, y > 0.$$

Proof. Assume (i). By Theorem 1 and remark 3, its generator f is of the form

$$f(x) = bxa^x, \quad x > 0,$$

for some $a, b \in (0, \infty)$. Since B_f is reflexive, that is $B_f(x, x) = x$ for all $x \in (0, \infty)$. Substituting here $x = 2$ and using Theorem 1 (ii), yields

$$2 = B_f(2, 2) = \frac{f(2)f(2)}{f(2+2)} = \frac{b \cdot 2 \cdot 2}{2+2} = b,$$

whence we get (3.1), which proves (ii).

Assume (ii). From (3.1) and the definition of B_f we get (iii).

The implication (iii) \implies (i) is obvious. \square

Because every homogeneous quasi-arithmetic mean is a power mean ([2], p. 249), our result implies the following

Corollary 2. *A homogeneous beta-type function is a quasi-arithmetic mean if, and only if, it is the harmonic mean.*

For another result connecting harmonic mean and the Euler Gamma function see [3].

4. CAUCHY DIFFERENCES AND A COROLLARY

Applying our main result, we obtain the following

Corollary 3. *Let $g : (0, \infty) \rightarrow \mathbb{R}$ be an arbitrary continuous function and let $p \in \mathbb{R}$. The following conditions are equivalent:*

(i) *the Cauchy difference is $p \log t$ -homogeneous, that is*

$$(4.1) \quad C_g(tx, ty) = C_g(x, y) + p \log t, \quad x, y, t > 0;$$

(ii) *there exist $c, d \in \mathbb{R}$ such that*

$$g(x) = cx + d - p \log t, \quad x > 0$$

and

$$C_g(x, y) = \log \left(\frac{xy}{x+y} \right)^p - d, \quad x, y > 0.$$

Proof. Setting $f := \exp \circ g$, we observe that condition (i) is equivalent to

$$B_f(tx, ty) = t^{-p} B_f(x, y), \quad x, y, t > 0,$$

since, using the definition of beta-type function, we have, for all $x, y > 0$,

$$e^{g(tx)+g(ty)-g(t(x+y))} = t^{-p} e^{g(x)+g(y)-g(x+y)}.$$

Taking the logarithm of both sides, we indeed obtain

$$-C_g(tx, ty) = \log t^{-p} - C_g(x, y), \quad x, y > 0,$$

and thus g satisfies (4.1).

By Theorem 1, there exist $a, b > 0$

$$f(x) = bx^{-p}a^x, \quad x > 0.$$

Thus, by the definition of f , we get, for all $x > 0$,

$$g(x) = \log b + p \log x + x \log a;$$

whence, putting $c := \log a$ and $d := \log b$, we obtain,

$$g(x) = cx + d + p \log x, \quad x > 0,$$

and consequently, for all $x, y > 0$,

$$\begin{aligned} C_g(x, y) &= g(x+y) - g(x) - g(y) \\ &= \log \left(\frac{xy}{x+y} \right)^p - d, \end{aligned}$$

which proves the implication (i) \implies (ii).

The second implication is easy to verify. \square

In connection with Cauchy differences and harmonic mean, let us note that Heuvers result [4] (see also Kannappan [5], p. 31) can be reformulated as

Remark 4. *The Cauchy difference of a function $f : (0, \infty) \rightarrow \mathbb{R}$ satisfies the functional equation*

$$(4.2) \quad C_f(x, y) = f \left(\frac{2}{H(x, y)} \right), \quad x, y > 0$$

if, and only if, f is a logarithmic function, i.e.

$$f(xy) = f(x) + f(y), \quad x, y > 0.$$

REFERENCES

- [1] J. Aczél, *Lectures on Functional Equations and Their Applications*, Academic Press, New York and London, 1966.
- [2] J. Aczél, J. G. Dhombres, *Functional Equations in Several Variables*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1989.
- [3] H. Alzer, *A harmonic mean inequality for the Gamma function*, J. Comput. Appl. Math. 87 (1997), 195-198.
- [4] K. J. Heuvers, *Another logarithmic functional equation*, Aeq. Math., **58** (1999), 260-264.
- [5] P. Kannappan, *Functional Equations and Inequalities with Applications*, Springer Monographs in Mathematics, Springer, New York, 2009.
- [6] M. Kuczma, A. Gilányi, *An Introduction to the Theory of Functional Equations and Inequalities*, 2009, Birkhäuser Verlag AG, Basel – Boston – Berlin.
- [7] M. Himmel, J. Matkowski, *Directional convexity and characterizations of Beta and Gamma functions* (submitted).
- [8] J. Matkowski, *Convergence of iterates of pre-mean type mappings*, ESAIM: Proceedings and Surveys, ECIT 2012, Witold Jarczyk, Daniele Fournier-Prunaret, João Manuel Gonçalves Cabral, November 2014, Vol. 46, 196-228.

Current address: Faculty of Mathematics, Computer Science and Econometrics University of Zielona Góra, Szafrana 4A, PL 65-516 Zielona Góra, Poland

E-mail address: `himmel@mathematik.uni-mainz.de`

Current address: Faculty of Mathematics, Computer Science and Econometrics University of Zielona Góra, Szafrana 4A, PL 65-516 Zielona Góra, Poland

E-mail address: `J.Matkowski@wmie.uz.zgora.pl`